$$(18) \qquad \int x^{m-1}dxX^p = \frac{x^{m-n}X^{p+1}}{b(m+pn)} - \frac{a(m-n)}{b(m+pn)}\int x^{m-n-1}dxX^p$$

$$(19) \qquad \int x^{m-1}dxX^p = \frac{x^m X^p}{m} - \frac{bpn}{m}\int x^{m+n-1}dxX^{p-1}$$

$$(20) \qquad \int x^{m-1}dxX^p = \frac{x^{m-n}X^{p+1}}{bn(p+1)} - \frac{m-n}{bn(p+1)}\int x^{m-n-1}dxX^{p+1}$$

$$(21) \qquad \int x^{m-1}dxX^p = \frac{x^m X^p}{m+pn} + \frac{apn}{m+pn}\int x^{m-1}dxX^{p-1}$$

$$(22) \qquad \int x^{m-1}dxX^p = -\frac{x^m X^{p+1}}{an(p+1)} + \frac{m+pn+n}{an(p+1)}\int x^{m-1}dxX^{p+1}.$$

It should also be observed that equations (14), (15) and (16) may be obtained directly by following the general method by which (1) and (2) were derived, and this method will be found much more simple than that generally pursued in obtaining reduction formulae for binomials.*

*Erratum. On page 138, line 6, for "general reductions" read general relations.

USEFUL FORMULAE IN THE CALCULUS OF FINITE DIFFERENCES.

BY G. W. HILL.

The finding of the values of the differential coefficients of a function of a single variable and of the single and double integrals with respect to the independent variable, from special values of the function computed at equidistant intervals, is an operation very frequent in Planetary Astronomy. The following seems a simpler exposition of the matter than has hitherto been given :

Let y be a function of x computed for the series of values of x, ... $a-h$, a, $a+h$, $a+2h$, ...; and let the differences and first and second summed values of y be denoted thus,

....	....	....	....	....	....	....	....
$a-h$	$\Delta^{-2}y_{-1}$	$\Delta^{-1}y_{-\frac{3}{2}}$	y_{-1}	$\Delta y_{-\frac{3}{2}}$	$\Delta^2 y_{-1}$	$\Delta^3 y_{-\frac{3}{2}}$	....
a	$\Delta^{-2}y_0$	$\Delta^{-1}y_{-\frac{1}{2}}$	y_0	$\Delta y_{-\frac{1}{2}}$	$\Delta^2 y_0$	$\Delta^3 y_{-\frac{1}{2}}$	....
$a+h$	$\Delta^{-2}y_1$	$\Delta^{-1}y_{\frac{1}{2}}$	y_1	$\Delta y_{\frac{1}{2}}$	$\Delta^2 y_1$	$\Delta^3 y_{\frac{1}{2}}$	....
$a+2h$	$\Delta^{-2}y_2$	$\Delta^{-1}y_{\frac{3}{2}}$	y_2	$\Delta y_{\frac{3}{2}}$	$\Delta^2 y_2$	$\Delta^3 y_{\frac{3}{2}}$	....
....	....	....	....	....	....	....	....

With regard to the differences of odd orders let us adopt the general notation

$$\Delta^{2n+1}y_i = \tfrac{1}{2}\left(\Delta^{2n+1}y_{i-\frac{1}{2}} + \Delta^{2n+1}y_{i+\frac{1}{2}}\right),$$

n and i being integers. In this way the symbol Δ does not follow the law of indices as in the ordinary method of differences, that is we do not have in general

$$\Delta^n \Delta^{n'} = \Delta^{n+n'}.$$

Nevertheless it is evident the following relations hold :

$$\Delta^{2n}\Delta^{2n'} = \Delta^{2(n+n')}, \quad \Delta^{2n+1}\Delta^{2n'} = \Delta^{2(n+n')+1},$$

that is, the exponents are to be added except when both are odd.

For brevity writing D for $\frac{d}{dx}$, and e denoting the base of hyperbolic logarithms, the symbolical expression for Taylor's Theorem gives

$$y_{-1} = e^{-hD}y_0, \quad y_0 = y_0, y_1 = e^{hD}y_0.$$

Whence it is easy to see that

$$\Delta = \frac{e^{hD} - e^{-hD}}{2}, \quad \Delta^2 = e^{hD} + e^{-hD} - 2.$$

The last may be written

$$\Delta^2 = \left(e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}\right)^2.$$

Thus it is evident that we have in general,

$$\Delta^{2n} = \left(e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}\right)^{2n}, \quad \Delta^{2n+1} = \left(e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}\right)^{2n+1} \cdot \frac{e^{\frac{hD}{2}} + e^{-\frac{hD}{2}}}{2}.$$

Also

$$e^{\frac{hD}{2}} - e^{-\frac{hD}{2}} = \sqrt{\Delta^2}, \quad e^{\frac{hD}{2}} + e^{-\frac{hD}{2}} = \sqrt{4 + \Delta^2}.$$

Whence

$$h^2D^2 = 4\log^2\left(\frac{\Delta}{2} + \sqrt{1 + \tfrac{1}{4}\Delta^2}\right) = \left(\int \frac{d\Delta}{\sqrt{1 + \frac{1}{4}\Delta^2}}\right)^2,$$

the integral being taken so as to vanish with Δ. The operation denoted by the symbolical expression $\frac{\Delta}{D}$ is evidently a function of Δ^2, and we have

$$\frac{\varDelta}{hD} = \left(e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}\right). \quad \frac{e^{\frac{hD}{2}} + e^{-\frac{hD}{2}}}{2hD} = \frac{\varDelta\sqrt{1+\frac{1}{4}\varDelta^2}}{hD}$$

$$= \varDelta\sqrt{1+\tfrac{1}{4}\varDelta^2}\left(\int\frac{d\varDelta}{\sqrt{1+\frac{1}{4}\varDelta^2}}\right)^{-1}.$$

Whence

$$hD = \frac{1}{\sqrt{1+\frac{1}{4}\varDelta^2}}\int\frac{d\varDelta}{\sqrt{1+\frac{1}{4}\varDelta^2}}.$$

It is plain, then, in general, that we have the value of an even differential coefficient from the formula

$$D^{2n} = h^{-2n}\left(\int\frac{d\varDelta}{\sqrt{1+\frac{1}{4}\varDelta^2}}\right)^{2n},$$

and the value of an odd one from

$$D^{2n+1} = \frac{h^{-2n-1}}{\sqrt{1+\frac{1}{4}\varDelta^2}}\left(\int\frac{d\varDelta}{\sqrt{1+\frac{1}{4}\varDelta^2}}\right)^{2n+1}.$$

Differentiating the first of these with respect to $\varDelta$, we obtain

$$\frac{d\,D^{2n}}{d\varDelta} = \frac{2nh^{-2n}}{\sqrt{1+\frac{1}{4}\varDelta^2}}\left(\int\frac{d\varDelta}{\sqrt{1+\frac{1}{4}\varDelta^2}}\right)^{2n-1} = 2nh^{-1}D^{2h-1}.$$

Thus the value of an even differential coefficient can be obtained from that of the preceding differential coefficient by the very simple formula

$$D^{2n} = 2nh^{-1}\textstyle\int D^{2n-1}d\varDelta.$$

If we employed the preceding formula for D for expanding its value in powers of $\varDelta$, we should find it difficult to discover the law of the numerical coefficients, but by differentiating the value of D^{2n+1} with respect to $\varDelta$, we shall find that it satisfies the differential equation

$$[1+\tfrac{1}{4}\varDelta^2]\frac{dD^{2n+1}}{d\varDelta} + \frac{\varDelta}{4}D^{2n+1} = (2n+1)h^{-1}D^{2n},$$

which when $n = 0$, becomes

$$[1+\tfrac{1}{4}\varDelta^2]\frac{dD}{d\varDelta} + \frac{\varDelta}{4}D = h^{-1}.$$

If in this we suppose

$$D = h^{-1}\Sigma.A_n\varDelta^n,$$

we shall find that the coefficients A_n satisfy in general the relation

$$(n+2)A_{n+2} + \frac{n+1}{4}A_n = 0,$$

whence

$$A_{n+2} = -\frac{n+1}{4(n+2)}A_n.$$

And as we know that $A_1 = 1$, this suffices for obtaining all the coefficients in succession. In the general case if we put

$$D^{2n} = h^{-2n}\Sigma.A_i^{(2n)}\Delta^i, \quad D^{2n+1} = h^{-2n-1}.\Sigma A_i^{(2n+1)}\Delta^i,$$

the above differential equation gives the following relation between the coefficients :

$$A_{i+2}^{(2n+1)} = -\frac{i+1}{4(i+2)}A_i^{(2n+1)} + \frac{2n+1}{i+2}A_{i+1}^{(2n)},$$

by which all the coefficients in succession may be derived.

We have

$$D = h^{-1}\left(\Delta - \frac{1}{3}\frac{\Delta^3}{2} + \frac{1.2}{3.5}\frac{\Delta^5}{2^2} - \frac{1.2.3}{3.5.7}\frac{\Delta^7}{2^3} + \dots\right),$$

$$D^2 = h^{-2}\left(\Delta^2 - \frac{1}{3.2}\frac{\Delta^4}{2} + \frac{1.2}{3.5.3}\frac{\Delta^6}{2^2} - \frac{1.2.3}{3.5.7.4}\frac{\Delta^8}{2^3} + \dots\right),$$

where the law of the coefficients is readily seen. In the higher differential coefficients the fractions are more complex, we therefore content ourselves with writing the values thus,

$$D^3 = h^{-3}\left(\Delta^3 - \frac{1}{4}\Delta^5 + \frac{7}{120}\Delta^7 - \frac{41}{1024}\Delta^9 + \dots\right),$$

$$D^4 = h^{-4}\left(\Delta^4 - \frac{1}{6}\Delta^6 + \frac{7}{240}\Delta^8 - \frac{41}{7560}\Delta^{10} + \dots\right),$$

$$D^5 = h^{-5}\left(\Delta^5 - \frac{1}{3}\Delta^7 + \frac{13}{144}\Delta^9 - \dots\dots\dots\right),$$

$$D^6 = h^{-6}\left(\Delta^6 - \frac{1}{4}\Delta^8 + \frac{13}{240}\Delta^{10} - \dots\dots\dots\right).$$

The above expressions for D^{2n} and D^{2n+1} are equally applicable when n is negative; they then give the formulae to be used in mechanical quadratures, thus

$$D^{-1} = \frac{h}{\sqrt{1+\frac{1}{4}\Delta^2}}\left(\int\frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}\right)^{-1},$$

$$D^{-2} = h^2\left(\int\frac{d\Delta}{\sqrt{1+\frac{1}{4}\Delta^2}}\right)^{-2}.$$

If these expressions are expanded in powers of Δ, we obtain

$$D^{-1} = h\left(\Delta^{-1} - \frac{1}{12}\Delta + \frac{11}{720}\Delta^3 - \frac{191}{60480}\Delta^5 + \frac{2497}{3628800}\Delta^7 - \frac{14797}{95800320}\Delta^9 + \frac{92427157}{2615348736000}\Delta^{11} - \dots\dots\right),$$

$$D^{-2} = h^2\left(\Delta^{-2} + \frac{1}{12}\Delta^0 - \frac{1}{240}\Delta^2 + \frac{31}{60480}\Delta^4 - \frac{289}{3628800}\Delta^6\right.$$

$$+\frac{317}{22809600}\triangle^{8}-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big).$$

These are the expressions to be used in computing the values of the integrals $\int ydx$ and $\int\int ydx^2$. It must be noticed that $\triangle^{-1}$ virtually contains an arbitrary constant C, and $\triangle^{-2}$ an arbitrary expression $Cx + C'$. In fact the quantities in the columns to the left of that of the function y cannot be written until we know one quantity in each column. These constants C and C' are usually determined from the given values of $\int ydx$ and $\int\int ydx^2$ for $x = a$. If we denote them by D_0^{-1} and D_0^{-2} and if in general the subscript $(_0)$ denote values which obtain when $x = a$, it will be seen that

$$\triangle_0^{-1} = \frac{D_0^{-1}}{h} + \frac{1}{12}\triangle_0 - \frac{11}{720}\triangle_0^3 + \ldots\ldots$$

$$\triangle_0^{-2} = \frac{D_0^{-2}}{h^2} - \frac{1}{12}\triangle_0^0 + \frac{1}{240}\triangle_0^2 - \ldots\ldots$$

Having thus the sum and difference of the quantities $\triangle^{-1}y_{-\frac{1}{2}}$ and $\triangle^{-1}y_{\frac{1}{2}}$, it will be easy to get the quantities themselves.

In using the method of mechanical quadratures it is usual to multiply the values of y by h, if the single integral only is wanted, but by h^2 if the double is also to be obtained; in the last case then it is necessary to divide the results obtained by h in order to have the single integral.

These formulae appear to have been first obtained by Gauss (*Werke*, Vol. III, p. 328). Encke has given them in the Berlin *Jahrbuch* for 1838. For use they are much superior to the formula given by Laplace (*Mecanique Celeste*, Vol. IV, p. 207).

PROBLEM IN "CURVES OF PURSUIT," BY WERNER STILLE, MARINE, ILL.—A hunter is in the field with his dog. The hunter, standing at a point A *calls* his dog, which is at a point B. At this moment the hunter walks off in a direction right-angular to AB, while the dog, running n times as fast as his master walks, keeps his master continually in view, thus at each moment running along the right line which then lies between him and his master.

Find thn Curve along which the dog runs until he overtakes his master at C.